Fourier Slice Theorem

1D F.T. of $g(p,\theta)$:

$$G(\omega,\theta) = \int g(p,\theta)\, e^{-2\pi i \omega p}\, dp$$

Frequencies in $g(p,\theta)$

θ fixed

$$G(\omega,\theta) = \int \overbrace{\iint f(x,y)\, \delta(x\cos\theta + y\sin\theta - p)\, dx\, dy}\, e^{-2\pi i \omega p}\, dp$$

$$= \iint f(x,y) \left[\int \delta(x\cos\theta + y\sin\theta - p)\, e^{-2\pi i \omega p}\, dp \right] dx\, dy$$

$$= \iint f(x,y)\, e^{-2\pi i \omega (x\cos\theta + y\sin\theta)}\, dx\, dy$$

$$= \left[\iint f(x,y)\, e^{-2\pi i (xu + yv)}\, dx\, dy \right]_{\substack{u = \omega\cos\theta \\ v = \omega\sin\theta}}$$

$$= \left[F(u,v) \right]_{\substack{u = \omega\cos\theta \\ v = \omega\sin\theta}}$$

SO THE 1D F.T. OF $g(p, \theta)$ IS THE SAME AS THE 2D F.T. OF $f(x,y)$ EVALUATED AT $(\omega \cos\theta, \omega \sin\theta)$

$$G(\omega, \theta) = \Big[F(u,v) \Big]_{\substack{u = \omega\cos\theta \\ v = \omega \sin\theta}}$$

$$= F(\omega\cos\theta, \omega\sin\theta)$$

FOR FIXED θ, $F(\omega\cos\theta, \omega\sin\theta)$

IS A LINE IN F IN DIRECTION $(\cos\theta, \sin\theta)$

PASSING THROUGH $F(0,0)$

[FIG 5.41]

FILTERED BACKPROJECTION

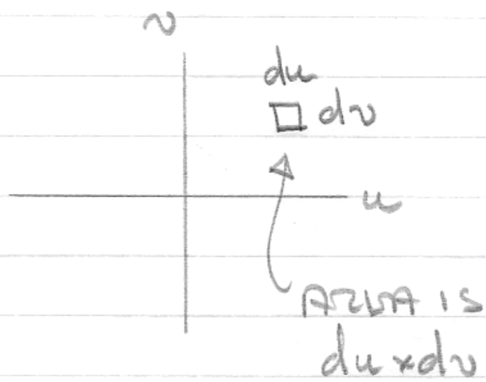
PLAIN BACKPROJECTION DIDN'T WORK WELL
(DUE TO HALOS)

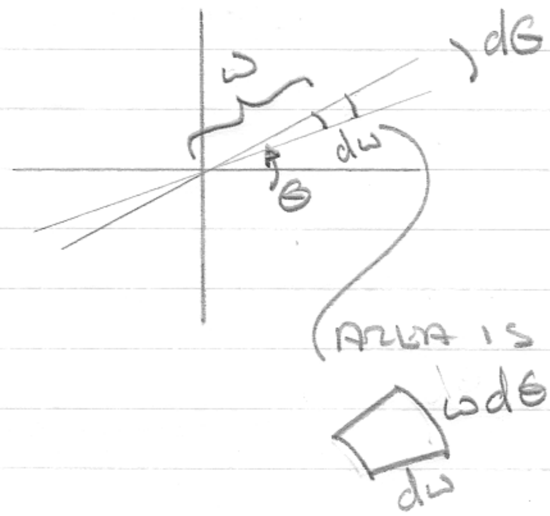
[FIG 5.39, 5.40]

IDEA: USE INVERSE F.T. TO RECONSTRUCT
$f(x,y)$

$$f(x,y) = \int_v \int_u F(u,v)\, e^{2\pi i(ux+vy)}\, du\, dv$$

REPARAMETRIZE THE u,v PLANE OF F
TO USE θ AND ω





$$\int_v \int_u \sim\sim\, du\, dv$$

$$\int_\theta \int_\omega \sim\sim\, |\omega|\, d\omega\, d\theta$$

↑ IN CASE $\omega < 0$

AND $(u,v) = (\omega \cos\theta, \omega \sin\theta)$

$$f(x,y) = \int_\theta \int_\omega F(\omega\cos\theta, \omega\sin\theta)\, e^{2\pi i\omega(x\cos\theta + y\sin\theta)} |\omega| d\omega d\theta$$

FOURIER SLICE THEOREM

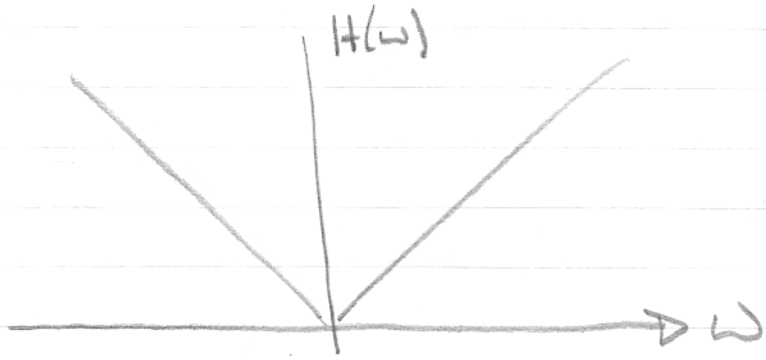
$$= \int_\theta \int_\omega G(\omega, \theta)\, e^{2\pi i\omega(x\cos\theta + y\sin\theta)} |\omega| d\omega d\theta$$

$$= \int_{\theta=0}^{\pi} \underbrace{\left[\int_\omega |\omega|\, G(\omega,\theta)\, e^{2\pi i\omega\rho}\, d\omega \right]}_{} d\theta$$

$\rho = x\cos\theta + y\sin\theta$

THIS IS G MULTIPLIED BY A FILTER, $H(\omega) = |\omega|$

$|\omega|$ is



WE COULD INSTEAD <u>CONVOLVE</u> THE INVERSE F.T. OF H WITH THE INVERSE F.T. OF G:

$$\int_{\omega} H(\omega)\, G(\omega,\theta)\, e^{2\pi i \omega p}\, d\omega \quad {\scriptstyle p = x\cos\theta + y\sin\theta}$$

$$= h(p) * g(p,\theta) \quad {\scriptstyle p = x\cos\theta + y\sin\theta}$$

$$\text{So } f(x,y) = \int_{\theta=0}^{\pi} h(p) * g(p,\theta)\, d\theta$$

$$p = x\cos\theta + y\sin\theta$$

FILTERED BACKPROJECTION

STEPS TO RECONSTRUCT $f(x,y)$

FOR EACH θ_k IN $[0, \pi)$:

- USE CT TO FIND $g(\rho, \theta_k)$
- COMPUTE $h(\rho) * g(\rho, \theta_k) = (h*g)_k(\rho)$

FOR EACH (x,y):

- $f(x,y) = \sum_{\theta_k} (h*g)_k(x\cos\theta_k + y\sin\theta_k)$

EQUIVALENT TO 'SMEARING' EACH $(h*g)_k$ ACROSS THE WHOLE IMAGE

FILTER $H(\omega)$

$$H(\omega) = |\omega|$$

"RAM-LAK FILTER"

(RAMACHANDRAN AND LAKSHMINARAYANAN)

SHARP DROPOFF AT EDGES RESULTS IN RINGING

$$H(\omega) = c + (c-1)\cos\frac{2\pi\omega}{R}$$

"HAMMING FILTER"

OTHER FILTERS EXIST

[FIG 5.42]

[SEE APPLET

BIGWWW.EPFL.CH/DEMO/JTOMOGRAPHY/DEMO.HTML]

FAN-BEAM CT

[FIG 5.45]

- DERIVATION IS MORE COMPLICATED

- COULD CONVERT TO A PARALLEL-BEAM PROBLEM, BUT SAMPLING IS NOT REGULAR.

CONE-BEAM CT

- EVEN WORSE DERIVATION
- APPROXIMATIONS USED, BUT RECONSTRUCTION IS GOOD

SPIRAL CT

- NEVER HAVE $g(p,\theta)$ IN SAME Z PLANE

- MUST INTERPOLATE IN Z